$$
\Phi = \begin{bmatrix} 0.5151 & 1.0883 & 1.3861 & 1.2653 & 0.7512 \\ 0.9435 & 1.3207 & 0.3001 & -1.0944 & -1.2705 \\ 1.2128 & 0.5144 & -1.3212 & -0.3187 & 1.3976 \\ 1.2778 & -0.6965 & -0.5861 & 1.3701 & -1.0934 \\ 1.1275 & -1.3596 & 1.1943 & -0.8664 & 0.4517 \end{bmatrix}
$$

$$
\Phi^T \Phi = \begin{bmatrix} 5.5305 & 0.0076 & -0.0076 & 0.0066 & -0.0045 \\ 0.0076 & 5.5266 & 0.0098 & -0.0086 & 0.0059 \\ -0.0076 & 0.0098 & 5.5266 & 0.0088 & -0.0063 \\ 0.0066 & -0.0086 & 0.0088 & 5.5282 & 0.0061 \\ -0.0045 & 0.0059 & -0.0063 & 0.0061 & 5.5314 \end{bmatrix}
$$

So the columns of Φ are not orthogonal.

Summary

The conjecture about orthogonal functions proposed in Ref. 1 has been shown to be implied by a classical theorem in the case of orthogonal polynomials and to be not necessarily true for other orthogonal functions.

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Reply by the Author to S. D. Hendry

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IN "Conjecture About Orthogonal Functions,"[1] I presented a conjecture "...to provoke insight that could lead to a proof of the conjecture at a later date." That insight came from Stephen Hendry in his comment about that paper. Hendry showed that the conjecture follows from a theorem given in the 1939 textbook entitled *Orthogonal Polynomials* by Szego.[2] The theorem is called the Gauss–Jacobi quadrature theorem, although contributions to the theorem were also made by Christoffel, Mehler, and Heine.[3–8] The theorem is stated as follows.

If $x_1 < x_2 < \cdots < x_n$ denote the zeros of the polynomial $p_n(x)$, there exist real numbers $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ such that

$$
\int_a^b \rho(x) w(x)\, \mathrm{d}(x) = \Lambda_1 \rho(x_1) + \Lambda_2 \rho(x_2) + \cdots \Lambda_n \rho(x_n) \tag{1}
$$

The function $\rho(x)$ is a linear combination of the ordered orthogonal polynomials $p_0(x), p_1(x), \ldots, p_{2n-1}(x)$, and $w(x)$ is the associated weighting function.[2]

The coefficients $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$, which are called Christoffel numbers, are associated with a set of orthogonal polynomials and the number n of orthogonal polynomials. This remarkable theorem produces an exact quadrature of functions contained in a $2n$-dimensional space by summing n terms regardless of the functions in that space!

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As Hendry stated, letting $\rho(x) = p_r(x) p_s(x)$ $(r = s = 1, 2, \ldots, n)$ in Eq. (1) reduces Eq. (1) to a transformation between the orthogonality condition for functions and the orthogonality condition for n-dimensional vectors, which is precisely the conjecture in Ref. 1. It was this conjecture that was used to develop a method of controlling transient vibration in distributed systems.[1]

Hendry also pointed out for a particular orthogonal set of sinusoidal functions treated in Ref. 1 that the conjecture was violated. It should also be pointed out that, whereas the conjecture was violated for that set of orthogonal functions, the conjecture was still accurate to three decimal places, indicating an additional strength of the Gauss–Jacobi quadrature theorem—that as a practical matter it can be applied to more than orthogonal polynomials.

While on the topic of the Gauss–Jacobi quadrature theorem, it is interesting to observe that letting $\rho(x) = p_r(x) w(x, t)$ in Eq. (1) reduces it to a quadrature of the rth modal coordinate, i.e., a modal filter.[9] The quadrature is exact if $w(x, t)$ is contained in the $(2n-r)$-dimensional space generated by the orthogonal polynomials.

Finally, the proof of the Gauss–Jacobi quadrature theorem, Eq. (1), is given in the Appendix for the reader's benefit.[2]

Appendix: Proof of Gauss–Jacobi Quadrature Theorem

Equation (1) is now derived. As preliminaries, we obtain the orthogonal polynomials $p_0(x), p_1(x), \ldots, p_n(x)$ by orthogonalizing $1, x, x^2, \ldots, x^n$. They are unique provided $p_r(x)$ is a polynomial of precise degree r in which the coefficient of x^r is positive and provided that

$$
\int_a^b p_r(x) p_s(x) w(x)\, \mathrm{d}(x) = \delta_{rs} \qquad (r, s = 1, 2, \ldots, n)
$$

We let $w(x)$ represent a weighting function, assumed to be nonnegative and measurable in the Lebesgue's sense and such that

$$
\int_a^b w(x)\, \mathrm{d}(x) > 0
$$

Depending on the limits of integration and on the weighting function, we obtain such classical orthogonal polynomials as Jacobi, Laguerre, Hermite, ultraspherical, Tchebichef of the first and second kinds, and Legendre.

A function in the linear space generated by the orthogonal polynomials up to order n is said to be contained in π_n. Notice that

$$
\int_a^b p_r(x) \rho(x) w(x)\, \mathrm{d}x = 0
$$

if $\rho(x)$ is in π_{r-1}, from which it follows that

$$
\int_a^b p_r(x) x^s w(x)\, \mathrm{d}x = 0 \qquad (s = 1, 2, \ldots, r-1)
$$

Next, notice that the orthogonal polynomials satisfy the Christoffel–Darboux recursive formula $p_r(x) = (A_r x + B_r) p_{r-1}(x) - C_r p_{r-2}(x)$ $(r = 2, 3, \ldots, n)$, in which $A_r = k_r / k_{r-1} > 0$ and $C_r = a_r / a_{r-1} = k_r k_{r-2} / k_{r-1}^2 > 0$ and where k_r is the highest coefficient of $p_r(x)$. To prove this, first determine A_r so that $p_r(x) - A_r x p_{r-1}(x)$ lies in π_{r-1}. Then look at

$$
0 = \int_a^b p_r(x) p_{r-2}(x) w(x)\, \mathrm{d}(x)
$$

to obtain C_r. Next, consider the important identity of the form $p_0(x) p_0(y) + p_1(x) p_1(y) + \cdots + p_n(x) p_n(y) = (k_n / k_{n+1}) [p_{n+1}(x) p_n(y) - p_x(x) p_{n+1}(y)]/(x - y)$, which follows from the Christoffel–Darboux recursive formula. Also, when $x = y$, notice that this important identity reduces to $p_0^2(x) + p_1^2(x) + \cdots + p_n^2(x) = (k_n / k_{n+1}) [p'_{n+1}(x) p_n(x) - p'_n(x) p_{n+1}(x)]$.

Now we approximate any $\rho(x)$ in π_{n-1} using the Lagrange interpolation polynomial of degree $n - 1$, written

$$L(x) = \sum_{v=1}^{n} \frac{\rho(x_v) p_n(x)}{p'_n(x_v)(x - x_v)}$$

Notice that $p_n(x)/[p'_n(x_v)(x - x_v)] = 1$ as x approaches x_v and $p_n(x_u)/[p_n(x_v)(x_u - x_v)] = 0$. It follows that $L(x_v) = \rho(x_v)$ $(v = 1, 2, \ldots, n)$.

The Gauss–Jacobi quadrature theorem is now proven by observing that the zeros of $\rho(x) - L(x)$ are $x_1, x_2, \ldots, x_n$, implying that it is a product of $p_n(x)$, that is, $\rho(x) - L(x) = p_n(x) r(x)$ for some $r(x)$ in π_{n-1}. Then

$$\int_a^b \rho(x) w(x)\,\mathrm{d}x = \int_a^b L(x) w(x)\,\mathrm{d}x = \sum_{v=1}^{n} \Lambda_v \rho(x_v)$$

in which

$$\Lambda_v = \int_a^b \frac{p_n(x)}{p'_n(x_v)(x - x_v)} w(x)\,\mathrm{d}x \qquad (v = 1, 2, \ldots, n) \qquad \square$$

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